

Modeling and Trajectory Design for Mechanical Control Systems

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Thanks to: Jorge Cortés, Andrew D. Lewis, Kevin Lynch, Sonia Martínez

Geometric Control of Mechanical Systems

Scientific Interests

- (i) success in linear control theory is unlikely to be repeated for nonlinear systems.
In particular, nonlinear system design. no hope for general theory
  mechanical systems as examples of control systems
- (ii) nonlinear control and geometric mechanics

Framework based on affine connections

- (i) reduction from $2n$ to n dimensional computations
- (ii) controllability, kinematic models, planning, averaging not stabilization

Literature review

Modeling:

- (i) R. Hermann. *Differential Geometry and the Calculus of Variations*, volume 49 of *Mathematics in Science and Engineering*. Academic Press, New York, NY, 1968
- (ii) A. M. Bloch and P. E. Crouch. Nonholonomic control systems on Riemannian manifolds. *SIAM JCO*, 33(1):126–148, 1995
- (iii) A. D. Lewis. Simple mechanical control systems with constraints. *IEEE T. Automatic Ctrl*, 45(8):1420–1436, 2000

Reductions & Planning via Inverse Kinematics:

- (i) H. Arai, K. Tanie, and N. Shiroma. Nonholonomic control of a three-DOF planar underactuated manipulator. *IEEE T. Robotics Automation*, 14(5):681–695, 1998
- (ii) K. M. Lynch, N. Shiroma, H. Arai, and K. Tanie. Collision-free trajectory planning for a 3-DOF robot with a passive joint. *Int. J. Robotic Research*, 19(12):1171–1184, 2000
- (iii) A. D. Lewis. When is a mechanical control system kinematic? In *Proc CDC*, pages 1162–1167, Phoenix, AZ, December 1999

Controllability:

- (i) H. J. Sussmann. A general theorem on local controllability. *SIAM JCO*, 25(1):158–194, 1987
- (ii) A. D. Lewis and R. M. Murray. Configuration controllability of simple mechanical control systems. *SIAM JCO*, 35(3):766–790, 1997

Averaging:

- (i) J. Baillieul. Stable average motions of mechanical systems subject to periodic forcing. In M. J. Enos, editor, *Dynamics and Control of Mechanical Systems: The Falling Cat and Related Problems*, volume 1, pages 1–23. Field Institute Communications, 1993
- (ii) M. Levi. Geometry of Kapitsa's potentials. *Nonlinearity*, 11(5):1365–8, 1998

Planning via approximate inversion:

- (i) R. E. Bellman, J. Bentsman, and S. M. Meerkov. Vibrational control of nonlinear systems: Vibrational stabilization. *IEEE T. Automatic Ctrl*, 31(8):710–716, 1986
- (ii) W. Liu. An approximation algorithm for nonholonomic systems. *SIAM JCO*, 35(4):1328–1365, 1997

Francesco Bullo and Andrew D. Lewis

Geometric Control of Mechanical Systems

Modeling, Analysis, and Design for Simple Mechanical Control Systems

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– Monograph –

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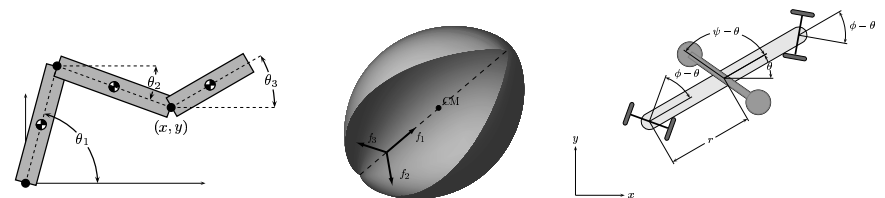
Outline: from geometry to algorithms

- (i) **modeling**
- (ii) **approach**
 - (a) analysis: kinematic reductions and controllability
 - (b) design: inverse kinematics catalog

Outline: from geometry to algorithms

- (i) **modeling**
- (ii) **approach #1**
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1 Models of Mechanical Control Systems



Ex #1: robotic manipulators with kinetic energy and forces at joints
systems with potential control forces

Ex #2: aerospace and underwater vehicles
invariant systems on Lie groups

Ex #3: systems subject to nonholonomic constraints
locomotion devices with drift, e.g., bicycle, snake-like robots

1.1 Basic geometric objects

- **manifold** $Q \subset \mathbb{R}^N$ $\mathbb{R}^n, T^n, S^n, SO(3), SE(3)$
- **vector fields** $X = (X^1, \dots, X^n) : Q \mapsto TQ$
- **metric** \mathbb{M} is an inner product on TQ and its inverse \mathbb{M}^{-1} matrix representations \mathbb{M}_{ij} and inverse \mathbb{M}^{lm}

- (i) a **connection** ∇ is a set of functions $\Gamma_{jk}^i : Q \rightarrow \mathbb{R}, i, j, k \in \{1, \dots, n\}$
- (ii) the **acceleration** of a curve $q : I \rightarrow Q$

$$(\nabla_{\dot{q}}\dot{q})^i = \ddot{q}^i + \Gamma_{jk}^i \dot{q}^j \dot{q}^k$$
- (iii) the **covariant derivative** $\nabla_X Y$ of two vector fields

$$(\nabla_X Y)^i = \frac{\partial Y^i}{\partial q^j} X^j + \Gamma_{jk}^i X^j Y^k \quad \langle X : Y \rangle = \nabla_X Y + \nabla_Y X$$

1.2 Constraints, distributions and kinematic modeling

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \rho \cos \phi \\ \rho \sin \phi \\ 0 \\ 1 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \omega$$

(unicycle dynamics, simplest wheeled robot dynamics)

$$\begin{bmatrix} \dot{x}_r \\ \dot{y}_r \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos \theta \\ \sin \theta \\ \frac{1}{\ell} \tan \phi \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \omega$$

1.3 SMCS with Constraints: definition

A **simple mechanical control system with constraints** is

- (i) an n -dimensional **configuration manifold** Q ,
- (ii) a metric \mathbb{M} on Q describing the **kinetic energy**,
- (iii) a function V on Q describing the **potential energy**,
- (iv) a **dissipative force** F_{diss} ,
- (v) a distribution \mathcal{D} of feasible velocities describing the **constraints**
- (vi) a set of m covector fields $\mathcal{F} = \{F^1, \dots, F^m\}$ defining the **control forces**

$$(Q, \mathbb{M}, V, F_{\text{diss}}, \mathcal{D}, \mathcal{F} = \{F^1, \dots, F^m\})$$

1.4 SMCS with Constraints: governing equations

Given $(Q, \mathbb{M}, V, F_{\text{diss}}, \mathcal{D}, \mathcal{F})$, there exists procedure:

$$\nabla_{\dot{q}}\dot{q} = Y_0(q) + R(q) + \sum_{a=1}^m Y_a(q)u_a \tag{1}$$

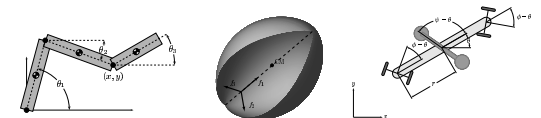
or, in coordinates:

$$\ddot{q}^k + \Gamma_{ij}^k(q)\dot{q}^i\dot{q}^j = Y_0(q)^k + R_i^k(q)\dot{q}^i + \sum_{a=1}^m Y_a^k(q)u_a$$

or, in different coordinates for the velocities,

$$\dot{q} = v^i X_i(q)$$

$$\dot{v}^k + \Gamma_{ij}^k(q)v^i v^j = Y_0(q)^k + R_i^k(q)v^i + \sum_{a=1}^m Y_a^k(q)u_a$$



1.5 Modeling construction

(Lewis, IEEE TAC '00)

From $(Q, \mathbb{M}, V, F_{\text{diss}}, \mathcal{D}, \mathcal{F})$ to

$$\nabla_{\dot{q}} \dot{q} = Y_0(q) + R(\dot{q}) + \sum_{a=1}^m Y_a(q) u_a$$

(i) $P: TQ \rightarrow TQ$ is the \mathbb{M} -orthogonal projection onto \mathcal{D}

(ii) $Y_0(q) = -P(\mathbb{M}^{-1}(dV))$

(iii) $R(\dot{q}) = P(\mathbb{M}^{-1}(F_{\text{diss}}(\dot{q})))$

(iv) $Y_a = P(\mathbb{M}^{-1}(F^a))$

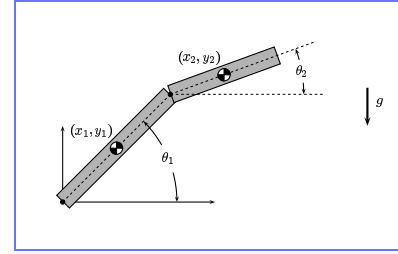
(v) ${}^{\mathbb{M}}\nabla$ is the **Levi-Civita connection** on (Q, \mathbb{M})

$$\Gamma_{ij}^k = \frac{1}{2} \mathbb{M}^{mk} \left(\frac{\partial \mathbb{M}_{mj}}{\partial q^i} + \frac{\partial \mathbb{M}_{mi}}{\partial q^j} - \frac{\partial \mathbb{M}_{ij}}{\partial q^m} \right) \quad (2)$$

(vi) ∇ is the **constrained affine connection** on $(Q, \mathbb{M}, \mathcal{D})$

$$\nabla_X Y = {}^{\mathbb{M}}\nabla_X Y - ({}^{\mathbb{M}}\nabla_X P)(Y) \quad (3)$$

1.6 Planar two links manipulator



$$(\theta_1, \theta_2) \in Q = \mathbb{T}^2$$

$$\mathbb{M} = \begin{bmatrix} I_1 + (l_1^2(m_1 + 4m_2))/4 & (l_1 l_2 m_2 \cos(\theta_1 - \theta_2))/2 \\ (l_1 l_2 m_2 \cos(\theta_1 - \theta_2))/2 & I_2 + (l_2^2 m_2)/4 \end{bmatrix}$$

$$V(\theta_1, \theta_2) = m_1 g l_1 \sin \theta_1 / 2 + m_2 g (l_1 \sin \theta_1 + l_2 / 2 \sin \theta_2)$$

no F_{diss}
no constraints
 $F^1 = d\theta_1, F^2 = d\theta_2 - d\theta_1$

Equations of motion:

$$\begin{pmatrix} \ddot{\theta}_1 & + \Gamma_{11}^1 \dot{\theta}_1 \dot{\theta}_1 + \Gamma_{12}^1 \dot{\theta}_1 \dot{\theta}_2 + \Gamma_{22}^1 \dot{\theta}_2 \dot{\theta}_2 \\ \ddot{\theta}_2 & + \Gamma_{11}^2 \dot{\theta}_1 \dot{\theta}_1 + \Gamma_{12}^2 \dot{\theta}_1 \dot{\theta}_2 + \Gamma_{22}^2 \dot{\theta}_2 \dot{\theta}_2 \end{pmatrix} = Y_0 + u_1 Y_1 + u_2 Y_2$$

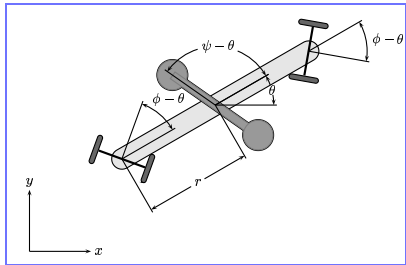
1.7 The snakeboard

$$(x, y, \theta, \psi, \phi) \in Q = SE(2) \times \mathbb{T}^2$$

$$F^1 = d\psi, F^2 = d\phi$$

$$\mathbb{M} = \begin{pmatrix} m & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 \\ 0 & 0 & \ell^2 m & J_r & 0 \\ 0 & 0 & J_r & J_r & 0 \\ 0 & 0 & 0 & 0 & J_w \end{pmatrix}$$

$$X_1 = \begin{pmatrix} \ell \cos \phi \cos \theta \\ \ell \cos \phi \sin \theta \\ -\sin \phi \\ 0 \\ 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$



$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\psi} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \ell \cos \phi \cos \theta \\ \ell \cos \phi \sin \theta \\ -\sin \phi \\ 0 \\ 0 \end{pmatrix} v_1 + \begin{pmatrix} \frac{J_r}{m\ell} \cos \phi \sin \phi \cos \theta \\ \frac{J_r}{m\ell} \cos \phi \sin \phi \sin \theta \\ -\frac{J_r}{m\ell^2} (\sin \phi)^2 \\ 1 \\ 0 \end{pmatrix} v_2 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} v_3$$

$$\dot{v}_1 + \frac{J_r}{m\ell^2} (\cos \phi) v_2 v_3 = 0$$

$$\dot{v}_2 - \frac{m\ell^2 \cos \phi}{m\ell^2 + J_r (\sin \phi)^2} v_1 v_3 - \frac{J_r \cos \phi \sin \phi}{m\ell^2 + J_r (\sin \phi)^2} v_2 v_3 = \frac{m\ell^2}{m\ell^2 J_r + J_r^2 (\sin \phi)^2} u_\psi$$

$$\dot{v}_3 = \frac{1}{J_w} u_\phi$$

$$\dot{q} = v^i X_i(q), \quad \dot{v}^k + ({}^X \Gamma_{ij}^k(q) v^i v^j) = Y_0(q)^k + R_i^k(q) \dot{q}^i + \sum_{a=1}^m Y_a^k(q) u_a$$

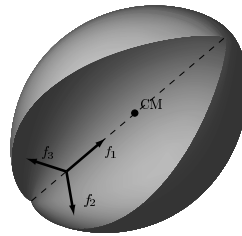
1.8 Underwater Vehicle in Ideal Fluid

3D rigid body with three forces:

(i) $(R, p) \in SE(3), (\Omega, V) \in \mathbb{R}^6$

(ii) $KE = \frac{1}{2}\Omega^T \mathbb{J}\Omega + \frac{1}{2}V^T \mathbb{M}V,$
 $\mathbb{M} = \text{diag}\{m_1, m_2, m_3\},$
 $\mathbb{J} = \text{diag}\{J_1, J_2, J_3\}$

(iii) $f_1 = e_4, f_2 = -he_3 + e_5, f_3 = he_2 + e_6$



Equations of Motion:

$$\begin{pmatrix} \dot{R} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} R\hat{\Omega} \\ RV \end{pmatrix}, \quad \begin{bmatrix} \mathbb{J}\dot{\Omega} - \mathbb{J}\Omega \times \Omega + \mathbb{M}V \times V \\ \mathbb{M}\dot{V} - \mathbb{M}V \times \Omega \end{bmatrix} = u_1 f_1 + u_2 f_2 + u_3 f_3$$

Outline: from geometry to algorithms

- (i) modeling
- (ii) approach #1
 - (a) analysis: kinematic reductions and controllability
 - (b) design: inverse kinematics catalog
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2 Analysis of Kinematic Reductions

Goal: (low-complexity) kinematic representations for mechanical control systems

Assume: no potential energy, no dissipation: $(Q, \mathbb{M}, V = 0, F_{\text{diss}} = 0, \mathcal{D}, \mathcal{F})$

(i) dynamic model with accelerations as control inputs mechanical systems:

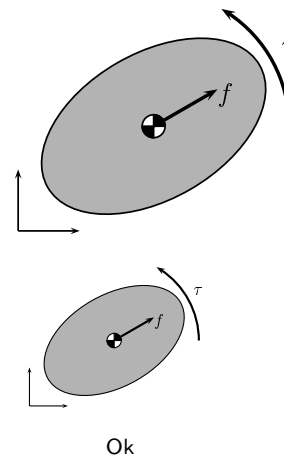
$$\nabla_{\dot{q}} \dot{q} = \sum_{a=1}^m Y_a(q) u_a(t) \quad \mathcal{V} = \text{span}\{Y_1, \dots, Y_m\}$$

(ii) kinematic model with velocities as control inputs

$$\dot{q} = \sum_{b=1}^{\ell} V_b(q) w_b(t) \quad \mathcal{V} = \text{span}\{V_1, \dots, V_{\ell}\}$$

ℓ is the rank of the reduction

2.1 When can a second order system follow the solution of a first order?



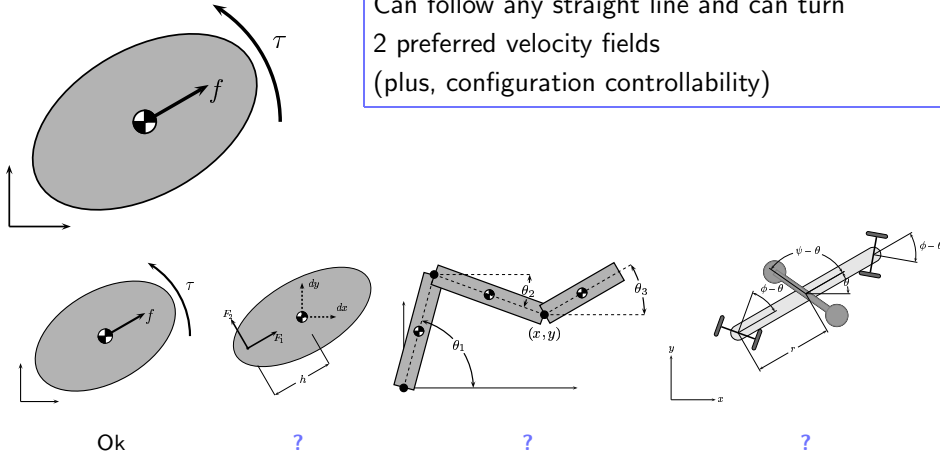
ex:

Can follow any straight line and can turn
 2 preferred velocity fields
 (plus, configuration controllability)

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ex:

Can follow any straight line and can turn
 2 preferred velocity fields
 (plus, configuration controllability)



2.2 Kinematic reductions

(Bullo and Lynch, IEEE TRA '01)

$\mathcal{V} = \text{span}\{V_1, \dots, V_\ell\}$ is a **kinematic reduction** if any curve $q: I \rightarrow Q$ solving the (controlled) kinematic model can be lifted to a solution of the (controlled) dynamic model.

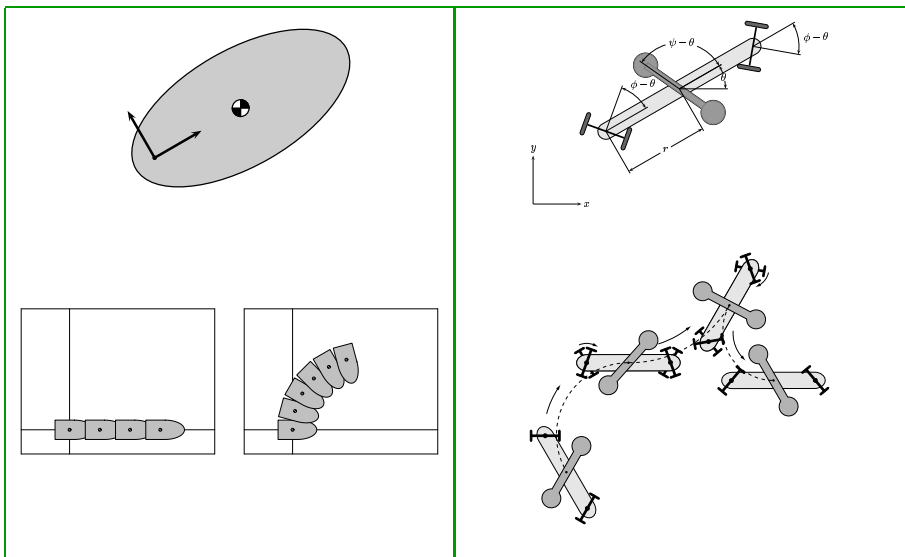
rank 1 reductions are called **decoupling vector fields**

Theorem The kinematic model induced by $\{V_1, \dots, V_\ell\}$ is a kinematic reduction of $(Q, \mathbb{M}, V=0, F_{\text{diss}}=0, \mathcal{D}, \mathcal{F})$ if and only if

- (i) $\mathcal{V} \subset \mathcal{Y}$
- (ii) $\langle \mathcal{V} : \mathcal{V} \rangle \subset \mathcal{Y}$

2.3 Examples of kinematic reductions

(Bullo and Lewis, IEEE TRA '03)



Two rank 1 kinematic reductions (decoupling vector fields)
 no rank 2 kinematic reductions

2.4 Examples of maximally reducible systems

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \rho \cos \phi \\ \rho \sin \phi \\ 0 \\ 1 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \omega$$

(unicycle dynamics, simplest wheeled robot dynamics)

$$\begin{bmatrix} \dot{x}_r \\ \dot{y}_r \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos \theta \\ \sin \theta \\ \frac{1}{\ell} \tan \phi \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \omega$$

2.5 When is a mechanical system kinematic?

(Lewis, CDC '99)

When are all dynamic trajectories executable by a single kinematic model?

A dynamic model is **maximally reducible (MR)** if all its controlled trajectory (starting from rest) are controlled trajectory of a single kinematic reduction.

Theorem $(Q, M, V=0, F_{diss}=0, \mathcal{D}, \mathcal{F})$ is maximally reducible if and only if

- (i) the kinematic reduction is the input distribution \mathcal{Y}
- (ii) $\langle \mathcal{Y} : \mathcal{Y} \rangle \subset \mathcal{Y}$

3 Controllability Analysis

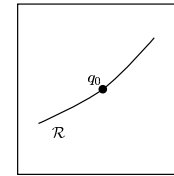
Objective: controllability notions and tests for mechanical systems and reductions

Assume: no potential energy, no dissipation: $(Q, M, V = 0, F_{diss} = 0, \mathcal{D}, \mathcal{F})$

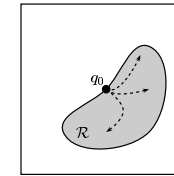
Review: Controllable kinematic systems

$$\dot{q} = \sum_{i=1}^{\ell} X_i(q)u_i(t)$$

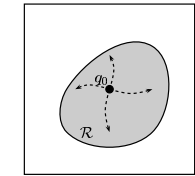
given two v.f.s X, Y , Lie bracket: $[X, Y]^k = \frac{\partial Y^k}{\partial q^i} X^i - \frac{\partial X^k}{\partial q^i} Y^i$ LARC



not accessible



accessible



controllable (STLC)

3.1 Controllability mechanisms

given control forces $\{F^1, \dots, F^m\}$



accessible accelerations $\{Y_1, \dots, Y_m\}$
 $Y_a = P(M^{-1}F^a)$



accessible velocities $\overline{\text{Sym}}\{Y_1, \dots, Y_m\}$
 $\{Y_i, \langle Y_j : Y_k \rangle, \langle \langle Y_j : Y_k \rangle : Y_h \rangle, \dots\}$



decoupling v.f.s $\{V_1, \dots, V_\ell\}$
 $V_i, \langle V_i : V_i \rangle \in \{Y_1, \dots, Y_m\}$



$\overline{\text{Lie}}\{V_1, \dots, V_\ell\}$: configurations accessible via decoupling v.f.s



access. configs $\overline{\text{Lie}}\{\overline{\text{Sym}}\{Y_1, \dots, Y_m\}\}$
 $\{Y_i, \langle Y_j : Y_k \rangle, [Y_j, Y_k], [\langle Y_j : Y_k \rangle, Y_h], \dots\}$



3.2 Controllability notions and tests

(Lewis and Murray, SIAM JCO '97)

V_1, \dots, V_ℓ decoupling v.f.s
 $\text{rank } \overline{\text{Lie}}\{V_1, \dots, V_\ell\} = n$



KC= locally kinematically controllable

$(q_0, 0) \xrightarrow{u} (q_f, 0)$ can reach open set of configurations by concatenating motions along kinematic reductions

$\text{rank } \overline{\text{Sym}}\{\mathcal{Y}\} = n$,
 "bad vs good"



STLC= small-time locally controllable

$(q_0, 0) \xrightarrow{u} (q_f, v_f)$ can reach open set of configurations and velocities

$\text{rank } \overline{\text{Lie}}\{\overline{\text{Sym}}\{\mathcal{Y}\}\} = n$,
 "bad vs good"

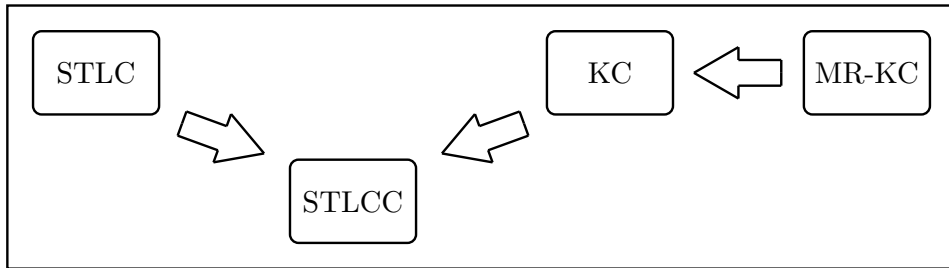


STLCC= small-time locally configuration controllable

$(q_0, 0) \xrightarrow{u} (q_f, v_f)$ can reach open set of configurations

3.3 Controllability inferences

- STLC = small-time locally controllable
- STLCC = small-time locally configuration controllable
- KC = locally kinematically controllable
- MR-KC = maximally reducible, locally kinematically controllable



There exist counter-examples for each missing implication sign.

3.4 Cataloging kinematic reductions and controllability of example systems

System	Picture	Reducibility	Controllability
planar 2R robot single torque at either joint: $(1, 0), (0, 1)$ $n = 2, m = 1$		$(1, 0)$: no reductions $(0, 1)$: maximally reducible	accessible not accessible or STLCC
roller racer single torque at joint $n = 4, m = 1$		no kinematic reductions	accessible, not STLCC
planar body with single force or torque $n = 3, m = 1$		decoupling v.f.	reducible, not accessible
planar body with single generalized force $n = 3, m = 1$		no kinematic reductions	accessible, not STLCC
planar body with two forces $n = 3, m = 2$		two decoupling v.f.	KC, STLC

robotic leg $n = 3, m = 2$		two decoupling v.f., maximally reducible	KC
planar 3R robot, two torques: $(0, 1, 1), (1, 0, 1), (1, 1, 0)$ $n = 3, m = 2$		$(1, 0, 1)$ and $(1, 1, 0)$: two decoupling v.f. $(0, 1, 1)$: two decoupling v.f. and maximally reducible	$(1, 0, 1)$ and $(1, 1, 0)$: KC and STLC $(0, 1, 1)$: KC
rolling penny $n = 4, m = 2$		fully reducible	KC
snakeboard $n = 5, m = 2$		two decoupling v.f.	KC, STLCC
3D vehicle with 3 generalized forces $n = 6, m = 3$		three decoupling v.f.	KC, STLC

Summary

- dynamic models (mechanics) vs kinematic models (trajectory analysis)
- general reductions (multiple, low rank) vs MR (one rank = m)
- STLCC (e.g., via STLC) vs kinematic controllability

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4 Trajectory Design via Inverse Kinematics

Objective: find u such that $(q_{\text{initial}}, 0) \xrightarrow{u} (q_{\text{target}}, 0)$

Assume:

- (i) $(Q, \mathbb{M}, V=0, F_{\text{diss}}=0, \mathcal{D}, \mathcal{F})$ is **kinematically controllable**

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- (ii) $Q = G$ and decoupling v.f.s $\{V_1, \dots, V_\ell\}$ are left-invariant
 \implies matrix exponential $\exp: \mathfrak{g} \rightarrow G$ gives closed-form flow

No general methodology is available \implies catalog for relevant example systems
 SO(3), SE(2), SE(3), etc

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 \implies matrix exponential $\exp: \mathfrak{g} \rightarrow G$ gives closed-form flow

Objective: select a finite-length combination of k flows along $\{V_1, \dots, V_\ell\}$ and coasting times $\{t_1, \dots, t_k\}$ such that

$$q_{\text{initial}}^{-1} q_{\text{target}} = g_{\text{desired}} = \exp(t_1 V_{i_1}) \cdots \exp(t_k V_{i_k}).$$

No general methodology is available \implies catalog for relevant example systems
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4.1 Inverse-kinematic planner on $SO(3)$ (Martínez, Cortés, and Bullo, IROS '03)

Any underactuated controllable system on $SO(3)$ is equivalent to

$$V_1 = e_z = (0, 0, 1) \quad V_2 = (a, b, c) \text{ with } a^2 + b^2 \neq 0$$

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Any underactuated controllable system on $SO(3)$ is equivalent to

$$V_1 = e_z = (0, 0, 1) \quad V_2 = (a, b, c) \text{ with } a^2 + b^2 \neq 0$$

Motion Algorithm: given $R \in SO(3)$, flow along (e_z, V_2, e_z) for coasting times

$$t_1 = \text{atan2}(w_1 R_{13} + w_2 R_{23}, -w_2 R_{13} + w_1 R_{23}) \quad t_2 = \arccos\left(\frac{R_{33} - c^2}{1 - c^2}\right)$$

$$t_3 = \text{atan2}(v_1 R_{31} + v_2 R_{32}, v_2 R_{31} - v_1 R_{32})$$

$$\text{where } z = \begin{bmatrix} 1 - \cos t_2 \\ \sin t_2 \end{bmatrix}, \quad \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} ac & b \\ cb & -a \end{bmatrix} z, \quad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} ac & -b \\ cb & a \end{bmatrix} z$$

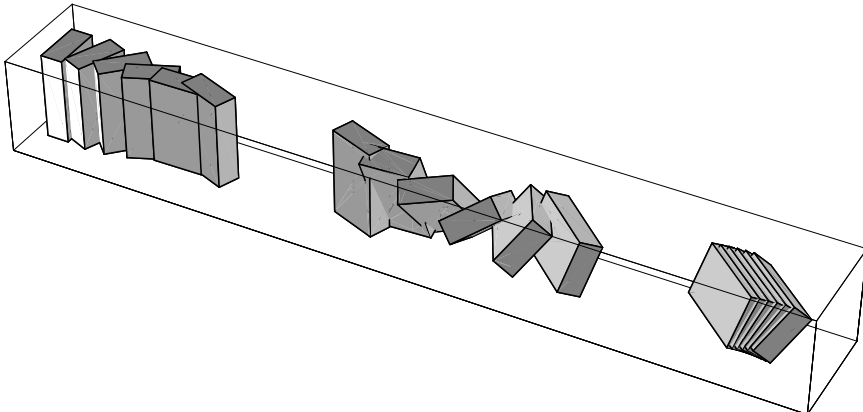
$$\text{Local Identity Map} = R \xrightarrow{IK} (t_1, t_2, t_3) \xrightarrow{FK} \exp(t_1 e_z) \exp(t_2 V_2) \exp(t_3 e_z)$$

4.2 Inverse-kinematic planner on $SO(3)$: simulation

The system can rotate about $(0, 0, 1)$ and $(a, b, c) = (0, 1, 1)$

Rotation from I_3 onto target rotation $\exp(\pi/3, \pi/3, 0)$

As time progresses, the body is translated along the inertial x -axis



4.3 Inverse-kinematic planner for Σ_1 -systems SE(2)

First class of underactuated controllable system on SE(2) is

$$\Sigma_1 = \{(V_1, V_2) \mid V_1 = (1, b_1, c_1), V_2 = (0, b_2, c_2), b_2^2 + c_2^2 = 1\}$$

Motion Algorithm: given (θ, x, y) , flow along (V_1, V_2, V_1) for coasting times

$$(t_1, t_2, t_3) = (\text{atan2}(\alpha, \beta), \rho, \theta - \text{atan2}(\alpha, \beta))$$

$$\text{where } \rho = \sqrt{\alpha^2 + \beta^2} \text{ and } \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} b_2 & c_2 \\ -c_2 & b_2 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} -c_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1 - \cos \theta \\ \sin \theta \end{bmatrix} \right)$$

$$\text{Identity Map} = (\theta, x, y) \xrightarrow{IK} (t_1, t_2, t_3) \xrightarrow{FK} \exp(t_1 V_1) \exp(t_2 V_2) \exp(t_3 V_1)$$

4.4 Inverse-kinematic planner for Σ_2 -systems SE(2)

Second and last class of underactuated controllable system on SE(2):

$$\Sigma_2 = \{(V_1, V_2) \mid V_1 = (1, b_1, c_1), V_2 = (1, b_2, c_2), b_1 \neq b_2 \text{ or } c_1 \neq c_2\}$$

Motion Algorithm: given (θ, x, y) , flow along (V_1, V_2, V_1) for coasting times

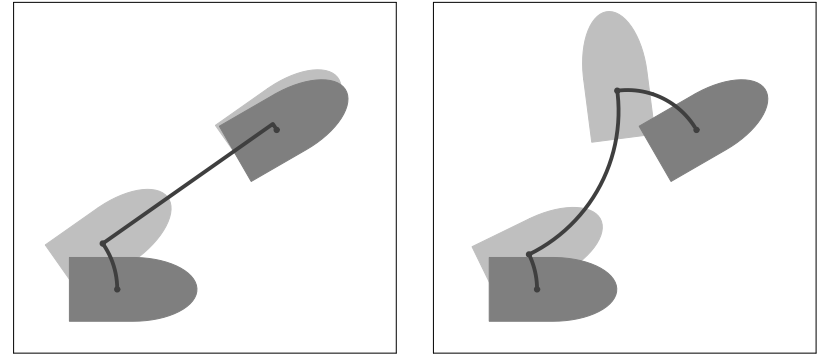
$$t_1 = \text{atan2}(\rho, \sqrt{4 - \rho^2}) + \text{atan2}(\alpha, \beta) \quad t_2 = \text{atan2}(2 - \rho^2, \rho\sqrt{4 - \rho^2})$$

$$t_3 = \theta - t_1 - t_2$$

where $\rho = \sqrt{\alpha^2 + \beta^2}$, $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} c_1 - c_2 & b_2 - b_1 \\ b_1 - b_2 & c_1 - c_2 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} -c_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1 - \cos \theta \\ \sin \theta \end{bmatrix} \right)$

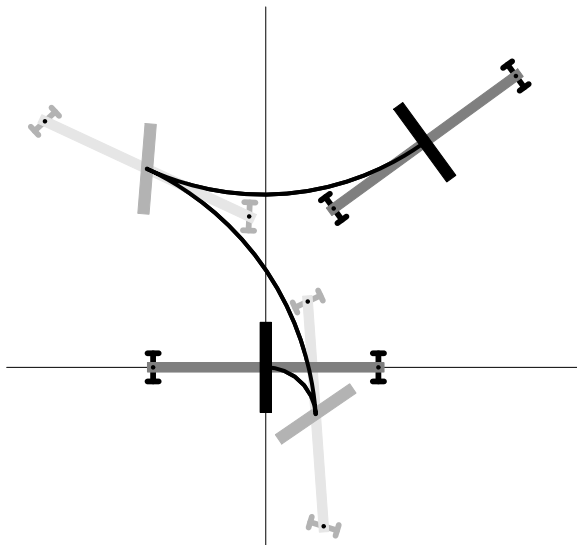
Local Identity Map = $(\theta, x, y) \xrightarrow{\mathcal{IK}} (t_1, t_2, t_3) \xrightarrow{\mathcal{FK}} \exp(t_1 V_1) \exp(t_2 V_2) \exp(t_3 V_1)$

4.5 Inverse-kinematic planners on SE(2): simulation



Inverse-kinematics planners for sample systems in Σ_1 and Σ_2 . The systems parameters are $(b_1, c_1) = (0, .5)$, $(b_2, c_2) = (1, 0)$. The target location is $(\pi/6, 1, 1)$.

4.6 Inverse-kinematic planners on SE(2): snakeboard simulation



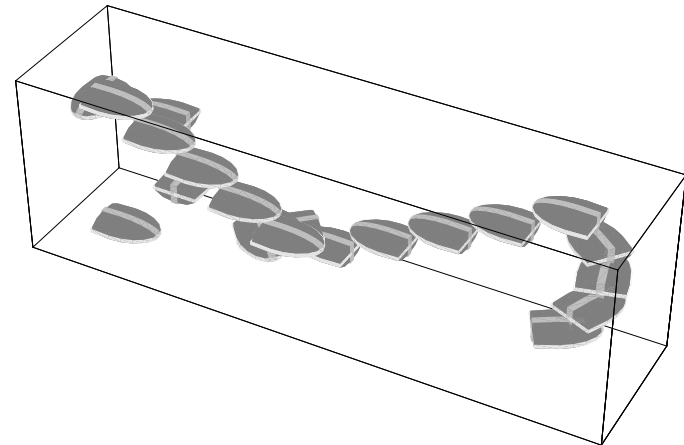
snakeboard as Σ_2 -system

4.7 Inverse-kinematic planners on SE(2) x R: simulation

4 dof system in \mathbb{R}^3 , no pitch no roll

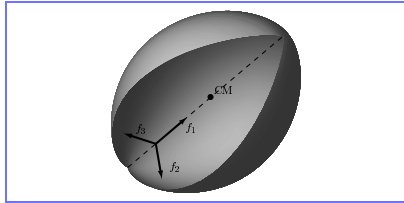
kinematically controllable via body-fixed constant velocity fields:

V_1 = rise and rotate about inertial point; V_2 = translate forward and dive

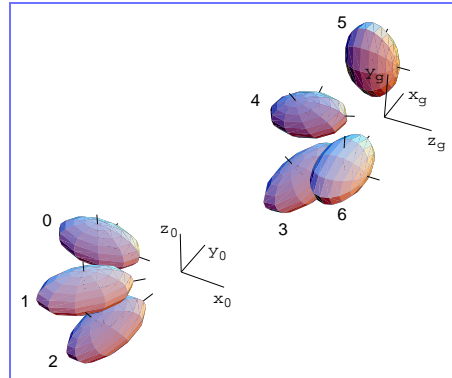


The target location is $(\pi/6, 10, 0, 1)$

4.8 Inverse-kinematic planners on SE(3): simulation



kinematically controllable via
body-fixed constant velocity fields:
 V_1 = translation along 1st axis
 V_2 = rotation about 2nd axis
 V_3 = rotation about 3rd axis



$V_3 : 0 \rightarrow 1$: rotation about 3rd axis
 $V_2 : 1 \rightarrow 2$: rotation about 2nd axis
 $V_1 : 2 \rightarrow 3$: translation along 1st axis
 $V_3 : 3 \rightarrow 4$: rotation about 3rd axis
 $V_2 : 4 \rightarrow 5$: rotation about 2nd axis
 $V_3 : 5 \rightarrow 6$: rotation about 3rd axis

Outline: from geometry to algorithms

(i) modeling and approach #1

- dynamic models (mechanics) vs kinematic models (trajectory analysis)
- general reductions (multiple, low rank) vs MR (one rank = m)
- STLCC (e.g., via STLC) vs kinematic controllability
- catalogs of systems and solutions

(ii) approach #2

- (a) analysis: oscillatory controls and averaging
- (b) design: approximate inversion

5 Averaging Analysis

Oscillations play key role in animal and robotic locomotion, oscillations generate motion in Lie bracket directions useful for trajectory design

Objective: oscillatory controls in mechanical systems

$$\nabla_{\dot{q}} \dot{q} = Y(q, t) \quad \int_0^T Y(q, t) dt = 0$$

5 Averaging Analysis

Oscillations play key role in animal and robotic locomotion, oscillations generate motion in Lie bracket directions useful for trajectory design

Objective: oscillatory controls in mechanical systems

$$\nabla_{\dot{q}} \dot{q} = Y(q, t) \quad \int_0^T Y(q, t) dt = 0$$

Assume: $(Q, M, V, F_{diss}, \mathcal{D}, \mathcal{F})$. Let $\epsilon > 0$

$$\nabla_{\dot{q}} \dot{q} = Y_0(q) + R(\dot{q}) + \sum_{a=1}^m \frac{1}{\epsilon} u_a \left(\frac{t}{\epsilon}, t \right) Y_a(q),$$

where u_a are T -periodic and zero-mean in their first argument.

5.1 Main Averaging Result

(Martínez, Cortés, and Bullo, IEEE TAC '03)

$$\nabla_{\dot{q}} \dot{q} = Y_0(q) + R(\dot{q}) + \sum_{a=1}^m \frac{1}{\epsilon} u_a \left(\frac{t}{\epsilon}, t \right) Y_a(q),$$



$$\nabla_{\dot{q}} \dot{q} = Y_0(q) + R(\dot{q}) - \sum_{a,b=1}^m \Lambda_{ab}(t) \langle Y_a : Y_b \rangle(q)$$

$$\Lambda_{ab}(t) = \frac{1}{2} \left(\bar{U}_{(a,b)}(t) + \bar{U}_{(b,a)}(t) - \bar{U}_{(a)}(t) \bar{U}_{(b)}(t) \right)$$

$$U_{(a)}(\tau, t) = \int_0^t u_a(\tau, s) ds, \quad U_{(a,b)}(\tau, t) = \int_0^t u_b(\tau, s_2) \int_0^{s_2} u_a(\tau, s_1) ds_1 ds_2$$

approximation valid over certain time scale

5.2 Averaging analysis with control potential forces

Assume no constraints ($\mathcal{D} = \text{TQ}$) and $\mathcal{F} = \{d\varphi_1, \dots, d\varphi_m\}$.

Then

$$Y_a(q) = \text{grad } \varphi_a(q), \quad (\text{grad } \varphi_a)^i = \mathbb{M}^{ij} \frac{\partial \varphi_a}{\partial q^j}$$

Symmetric product restricts

$$\langle \text{grad } \varphi_a : \text{grad } \varphi_b \rangle \equiv \text{grad } \langle \varphi_a : \varphi_b \rangle$$

where **Beltrami bracket (Crouch '81)**:

$$\langle \varphi_a : \varphi_b \rangle = \langle \langle d\varphi_a, d\varphi_b \rangle \rangle = \mathbb{M}^{ij} \frac{\partial \varphi_a}{\partial q^i} \frac{\partial \varphi_b}{\partial q^j}$$

5.3 Averaged potential

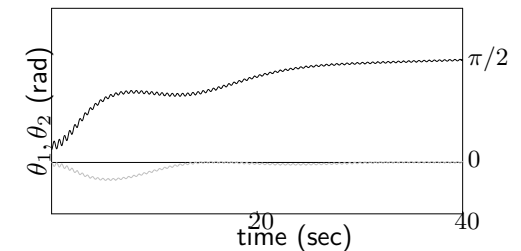
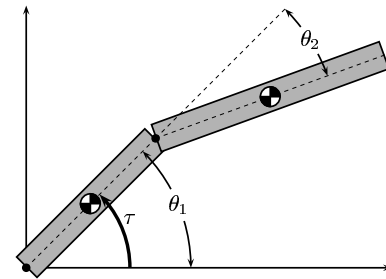
$$\mathbb{M} \nabla_{\dot{q}} \dot{q} = -\text{grad } V(q) + R(\dot{q}) + \sum_{a=1}^m u_a(t) \text{grad}(\varphi_a)(q).$$



$$\mathbb{M} \nabla_{\dot{q}} \dot{q} = -\text{grad } V_{\text{averaged}}(q) + R(\dot{q})$$

$$V_{\text{averaged}} = V + \sum_{a,b=1}^m \Lambda_{ab} \langle \varphi_a : \varphi_b \rangle$$

5.4 Oscillations stabilization example: a 2-link manipulator



$$u = \frac{1}{\epsilon} \cos \left(\frac{t}{\epsilon} \right)$$

Two-link damped manipulator with oscillatory control at first joint. The averaging analysis predicts the behavior. (the gray line is θ_1 , the black line is θ_2).

6 Trajectory Design via Oscillatory Controls and Approximate Inversion

Objective: steer configuration of $(Q, \mathbb{M}, V, F_{\text{diss}}, \mathcal{D}, \mathcal{F})$ along target trajectory $\gamma_{\text{target}}: [0, T] \rightarrow Q$ via oscillatory controls:

$$\nabla_{\dot{q}} \dot{q} = Y_0(q) + R(\dot{q}) + \sum_{a=1}^m u_a Y_a(q),$$

Low-order STLC assumption:

- (i) $\text{span}\{Y_a, \langle Y_b : Y_c \rangle \mid a, b, c \in \{1, \dots, m\}\}$ is full rank
- (ii) “bad vs good” condition: $\langle Y_a : Y_a \rangle \in \mathcal{Y} = \text{span}\{Y_a\}$.

6.1 From the STLC assumption ...

- (i) **fictitious inputs** $z_{\text{target}}^a, z_{\text{target}}^{ab}: [0, T] \rightarrow \mathbb{R}$, $a < b$, with

$$\begin{aligned} \nabla_{\gamma'_{\text{target}}} \gamma'_{\text{target}} &= Y_0(\gamma_{\text{target}}) + R(\gamma'_{\text{target}}) \\ &+ \sum_{a=1}^m z_{\text{target}}^a Y_a(\gamma_{\text{target}}(t)) + \sum_{a < b} z_{\text{target}}^{ab} \langle Y_a : Y_b \rangle(\gamma_{\text{target}}(t)), \end{aligned}$$

- (ii) for $a, b \in \{1, \dots, m\}$, **bad/good coefficient functions** $\alpha_{a,b}: Q \rightarrow \mathbb{R}$

$$\langle Y_a : Y_a \rangle = \sum_{b=1}^m \alpha_{a,b} Y_b.$$

6.1 From the STLC assumption ...

- (i) **fictitious inputs** $z_{\text{target}}^a, z_{\text{target}}^{ab}: [0, T] \rightarrow \mathbb{R}$, $a < b$, with

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- (ii) for $a, b \in \{1, \dots, m\}$, **bad/good coefficient functions** $\alpha_{a,b}: Q \rightarrow \mathbb{R}$

$$\langle Y_a : Y_a \rangle = \sum_{b=1}^m \alpha_{a,b} Y_b.$$

Also, there are $N = m(m-1)/2$ pairs of elements (a, b) in $\{1, \dots, m\}$, with $a < b$. Let $(a, b) \mapsto \omega(a, b) \in \{1, \dots, N\}$ be an enumeration of these pairs, and define **ω -frequency sinusoidal function**

$$\psi_{\omega(a,b)}(t) = \sqrt{2} \omega(a, b) \cos(\omega(a, b)t)$$

6.2 Trajectory tracking via Approximate Inversion

(Martínez, Cortés, and Bullo, IEEE TAC '03)

Theorem Consider $(Q, \mathbb{M}, V, F_{\text{diss}}, \mathcal{D}, \mathcal{F})$. Let

$$u_a = v_a(t, q) + \frac{1}{\epsilon} w_a \left(\frac{t}{\epsilon}, t \right)$$

with

$$w_a(\tau, t) =$$

$$v_a(t, q) =$$

Then, $t \mapsto q(t)$ follows γ_{target} with an error of order ϵ over the time scale 1.

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with

$$w_a(\tau, t) =$$

$$v_a(t, q) = z_{\text{target}}^a(t)$$

Then, $t \mapsto q(t)$ follows γ_{target} with an error of order ϵ over the time scale 1.

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with

$$w_a(\tau, t) = \sum_{c=a+1}^m z_{\text{target}}^{ac}(t) \psi_{\omega(a,c)}(\tau) - \sum_{c=1}^{a-1} \psi_{\omega(c,a)}(\tau)$$

$$v_a(t, q) = z_{\text{target}}^a(t)$$

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with

$$w_a(\tau, t) = \sum_{c=a+1}^m z_{\text{target}}^{ac}(t) \psi_{\omega(a,c)}(\tau) - \sum_{c=1}^{a-1} \psi_{\omega(c,a)}(\tau)$$

$$v_a(t, q) = z_{\text{target}}^a(t) + \frac{1}{2} \sum_{b=1}^m \alpha_{a,b}(q) \left(j-1 + \sum_{c=j+1}^m (z_{\text{target}}^{bc}(t))^2 \right)$$

Then, $t \mapsto q(t)$ follows γ_{target} with an error of order ϵ over the time scale 1.

6.3 Oscillatory controls ex. #1: A second-order nonholonomic integrator

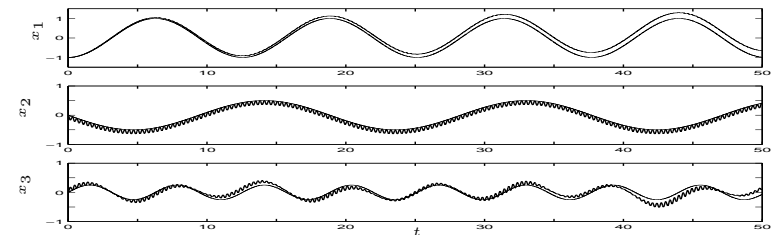
Consider

$$\ddot{x}_1 = u_1, \quad \ddot{x}_2 = u_2, \quad \ddot{x}_3 = u_1 x_2 + u_2 x_1,$$

Controllability assumption ok. Design controls to track $(x_1^d(t), x_2^d(t), x_3^d(t))$:

$$u_1 = \ddot{x}_1^d + \frac{1}{\sqrt{2}\epsilon} (\ddot{x}_3^d - \ddot{x}_1^d x_2^d - \ddot{x}_2^d x_1^d) \cos \left(\frac{t}{\epsilon} \right)$$

$$u_2 = \ddot{x}_2^d - \frac{\sqrt{2}}{\epsilon} \cos \left(\frac{t}{\epsilon} \right)$$



7 Summary: from geometry to algorithms

Trajectory design via kinematic reductions

- dynamic models (mechanics) vs kinematic models (trajectory analysis)
- general reductions (multiple, low rank) vs MR (one rank = m)
- STLCC (e.g., via STLC) vs kinematic controllability
- catalogs of systems and solutions

Trajectory design via averaging

- high-amplitude high-frequency two time-scales averaging
- general tracking result based on STLC assumption

trajectory analysis: reduction, controllability, averaging

trajectory design: inverse kinematics and approximate inversion

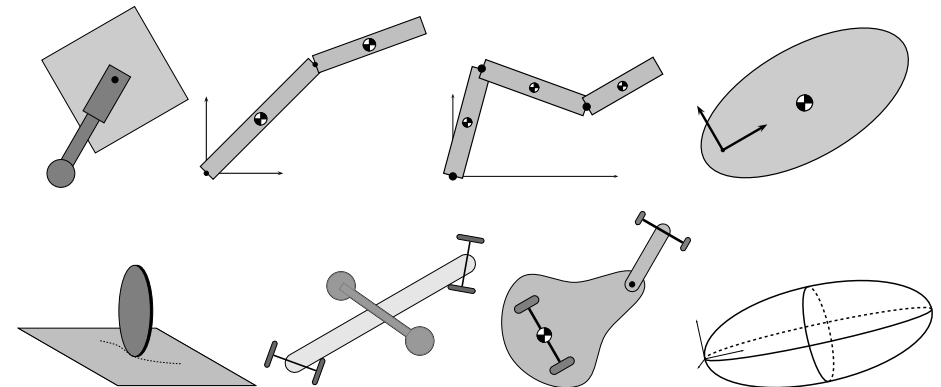
Future research

- weaken strict assumptions for reductions approach
 $V = 0$, kinematic controllability, group actions
- render second approach more realistic
- integrate with numerical and passivity methods for trajectory design
- locomotion in fluid (fishes, flying insects, etc)
- computational geometry and coordination in multi-vehicle systems

Research work reflected in this talk: (<http://motion.csl.uiuc.edu>)

- F. Bullo and M. Žefran. On mechanical control systems with nonholonomic constraints and symmetries. *IFAC Syst. & Control L.*, 45(2):133–143, 2002
- F. Bullo and K. M. Lynch. Kinematic controllability for decoupled trajectory planning in underactuated mechanical systems. *IEEE T. Robotics Automation*, 17(4):402–412, 2001
- F. Bullo, N. E. Leonard, and A. D. Lewis. Controllability and motion algorithms for underactuated Lagrangian systems on Lie groups. *IEEE T. Automatic Ctrl*, 45(8):1437–1454, 2000
- F. Bullo. Series expansions for the evolution of mechanical control systems. *SIAM JCO*, 40(1):166–190, 2001
- F. Bullo. Averaging and vibrational control of mechanical systems. *SIAM JCO*, 41(2):542–562, 2002
- S. Martínez, J. Cortés, and F. Bullo. Analysis and design of oscillatory control systems. *IEEE T. Automatic Ctrl*, 48(7):1164–1177, 2003
- F. Bullo and A. D. Lewis. Kinematic controllability and motion planning for the snakeboard. *IEEE T. Robotics Automation*, 19(3):494–498, 2003
- F. Bullo and A. D. Lewis. Low-order controllability and kinematic reductions for affine connection control systems. *SIAM JCO*, January 2004. To appear
- S. Martínez, J. Cortés, and F. Bullo. A catalog of inverse-kinematics planners for underactuated systems on matrix Lie groups. In *Proc IROS*, pages 625–630, Las Vegas, NV, October 2003
- F. Bullo. Trajectory design for mechanical systems: from geometry to algorithms. *European Journal of Control*, December 2003. Submitted

7.1 Examples



7.2 Comparison with perturbation methods for mechanical control systems

forced response of Lagrangian system from rest

I) High magnitude high frequency

"oscillatory control & vibrational stabilization"

$$H = H(q, p) + \frac{1}{\epsilon} \varphi \left(q, p, u \left(\frac{t}{\epsilon} \right) \right)$$

$$p(0) = p_0$$

II) Small input from rest

"small-time local controllability"

$$H = H(q, p) + \epsilon \varphi(q, p, u(t))$$

$$p(0) = 0$$

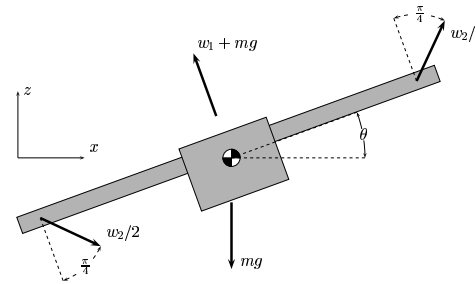
III) Classical formulation

integrable Hamiltonian systems

$$H = H(q, p) + \epsilon \varphi(q, p)$$

$$p(0) = p_0$$

7.3 A planar vertical takeoff and landing (PVTOL) aircraft



$$\dot{x} = \cos \theta v_x - \sin \theta v_z$$

$$\dot{z} = \sin \theta v_x + \cos \theta v_z$$

$$\dot{\theta} = \omega$$

$$\dot{v}_x - v_z \omega = -g \sin \theta + (-k_1/m)v_x + (1/m)u_2$$

$$\dot{v}_z + v_x \omega = -g(\cos \theta - 1) + (-k_2/m)v_z + (1/m)u_1$$

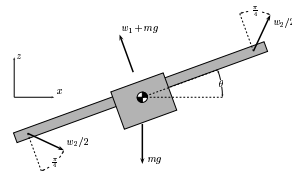
$$\dot{\omega} = (-k_3/J)\omega + (h/J)u_2$$

$Q = SE(2)$: Configuration and velocity space via $(x, z, \theta, v_x, v_z, \omega)$. x and z are horizontal and vertical displacement, θ is roll angle. The angular velocity is ω and the linear velocities in the body-fixed x (respectively z) axis are v_x (respectively v_z).

u_1 is body vertical force minus gravity, u_2 is force on the wingtips (with a net horizontal component). k_i -components are linear damping force, g is gravity constant. The constant h is the distance from the center of mass to the wingtip, m and J are mass and moment of inertia.

7.4 Oscillatory controls ex. #2: PVTOL model

Controllability assumption ok. Design controls to track $(x^d(t), z^d(t), \theta^d(t))$:



$$u_1 = \frac{J}{h} \ddot{\theta}^d + \frac{k_3}{h} \dot{\theta}^d - \frac{\sqrt{2}}{\epsilon} \cos \left(\frac{t}{\epsilon} \right)$$

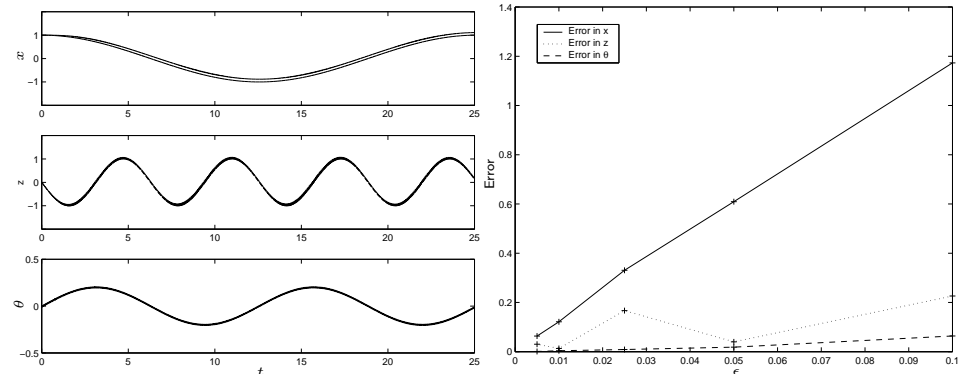
$$u_2 = \frac{h}{J} - f_1 \sin \theta^d + f_2 \cos \theta^d - \frac{J\sqrt{2}}{h\epsilon} (f_1 \cos \theta^d + f_2 \sin \theta^d) \cos \left(\frac{t}{\epsilon} \right),$$

where we let $c = \frac{J}{h} \ddot{\theta}^d + \frac{k_3}{h} \dot{\theta}^d$ and

$$f_1 = m\ddot{x}^d + (k_1 \cos^2 \theta^d + k_2 \sin^2 \theta^d) \dot{x}^d + \frac{\sin(2\theta^d)}{2} (k_1 - k_2) \dot{z}^d + mg \sin \theta^d - c \cos \theta^d,$$

$$f_2 = m\ddot{z}^d + \frac{\sin(2\theta^d)}{2} (k_1 - k_2) \dot{x}^d + (k_1 \sin^2 \theta^d + k_2 \cos^2 \theta^d) \dot{z}^d + mg(1 - \cos \theta^d) - c \sin \theta^d.$$

7.5 PVTOL Simulations: trajectories and error



Trajectory design at $\epsilon = .01$.

Tracking errors at $t = 10$.